Computers and Mathematics with Applications 62 (2011) 814–820

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Measure of noncompactness of matrix operators on some difference sequence spaces of weighted means

M. Mursaleen [a], V. Karakaya [b,*], H. Polat [c], N. Simşek [d]

[a] Aligarh Muslim University, Department of Mathematics, Aligarh 202002, India
[b] Yildiz Technical University, Department of Mathematical Engineering, Davutpasa Campus, Esenler, Istanbul, Turkey
[c] Muş Arparslan University, Department of Mathematics, Muş, Turkey
[d] Istanbul Commerce University, Department of Mathematics, Uskudar, Istanbul, Turkey

ARTICLE INFO

Article history:
Received 19 March 2011
Received in revised form 6 June 2011
Accepted 6 June 2011

Keywords:
Sequence space
Weighted mean
Matrix transformation
Compact operator
Hausdorff measure of noncompactness

ABSTRACT

For a sequence $x = (x_k)$, we denote the difference sequence by $\Delta x = (x_k - x_{k-1})$. Let $u = (u_k)_{k=0}^{\infty}$ and $v = (v_k)_{k=0}^{\infty}$ be the sequences of real numbers such that $u_k \neq 0$, $v_k \neq 0$ for all $k \in \mathbb{N}$. The difference sequence spaces of weighted means $\lambda(u, v, \Delta)$ are defined as

$$\lambda(u, v, \Delta) = \{x = (x_k) : W(x) \in \lambda\},$$

where $\lambda = c$, c_0 and ℓ_∞ and the matrix $W = (w_{nk})$ is defined by

$$w_{nk} = \begin{cases} u_n(v_k - v_{k+1}); & (k < n), \\ u_n v_n; & (k = n), \\ 0; & (k > n). \end{cases}$$

In this paper, we establish some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators on $\lambda(u, v, \Delta)$. Further, we characterize some classes of compact operators on these spaces by using the Hausdorff measure of noncompactness.

1. Introduction and preliminaries

For basic definitions and notation we refer to [1,2].

Let w denote the space of all real or complex sequences $x = (x_k)_{k=0}^{\infty}$. We write ϕ for the set of all finite sequences that terminate in zeros, $e = (1, 1, 1, \ldots)$ and $e^{(n)}$ for the sequence whose only non-zero term is 1 at the nth place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Also, we shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences respectively. Further, by cs and ℓ_1 we denote the spaces of all sequences associated with convergent and absolutely convergent series respectively.

The β-dual of a subset X of w is defined by

$$X^\beta = \left\{a = (a_k) \in w : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\right\}.$$

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix and A_n denote the sequence in the nth row of A, that is, $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. In addition, if $x = (x_k) \in w$ then we define the A-*transform* of x as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$,

* Corresponding author.
E-mail addresses: mursaleenm@gmail.com (M. Mursaleen), vkkaya@yildiz.edu.tr, vkkaya@yahoo.com (V. Karakaya), hpolat63@hotmail.com (H. Polat), necsimsek@yahoo.com (N. Simşek).

0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.06.011

where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k; \quad (n \in \mathbb{N}), \tag{1.1}$$

provided the series on the right converges for each $n \in \mathbb{N}$.

For arbitrary subsets X and Y of w, we write (X, Y) for the class of all infinite matrices that map X into Y. Thus $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$. Moreover, the *matrix domain* of an infinite matrix A in X is defined by

$$X_A = \left\{x \in w : Ax \in X\right\}.$$

The theory of *BK* spaces is the most powerful tool in the characterization of matrix transformations between sequence spaces.

A sequence space X is called a *BK space* if it is a Banach space with continuous coordinates $p_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $\mathbb{C}$ denotes the complex field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$.

The sequence spaces ℓ_∞, c and c_0 are *BK* spaces with the usual sup-norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$, where the supremum is taken over all $k \in \mathbb{N}$. Also, the space ℓ_1 is a *BK* space with the usual ℓ_1-norm defined by $\|x\|_{\ell_1} = \sum_{k=0}^{\infty} |x_k|$[3, Example 1.13].

Let X be a normed space. Then, we write S_X and $\bar{B}_X$ for the unit sphere and the closed unit ball in X, that is, $S_X = \{x \in X : \|x\| = 1\}$ and $\bar{B}_X = \{x \in X : \|x\| \leq 1\}$. If X and Y are *Banach spaces*, then $\mathcal{B}(X, Y)$ denotes the set of all bounded (continuous) linear operators $L : X \to Y$, which is a *Banach space* with the operator norm given by $\|L\| = \sup_{x \in S_X} \|L(x)\|_Y$ for all $L \in \mathcal{B}(X, Y)$. A linear operator $L : X \to Y$ is said to be *compact* if the domain of L is all of X and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a subsequence which converges in Y. We denote the class of all compact operators in $\mathcal{B}(X, Y)$ by $\mathcal{C}(X, Y)$. An operator $L \in \mathcal{B}(X, Y)$ is said to be *of finite rank* if $\dim R(L) < \infty$, where $R(L)$ is the range space of L. An operator of finite rank is clearly compact.

If $X \supset \phi$ is a *BK* space and $a = (a_k) \in w$, then we write

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right|, \tag{1.2}$$

provided the expression on the right is defined and finite which is the case whenever $a \in X^\beta$ [4, p. 381].

An infinite matrix $T = (t_{nk})$ is called a *triangle* if $t_{nn} \neq 0$ and $t_{nk} = 0$ for all $k > n$ $(n \in \mathbb{N})$. The study of matrix domains of triangles in sequence spaces has a special importance due to the various properties which they have. For example, if X is a *BK* space then X_T is also a *BK* space with the norm given by $\|x\|_{X_T} = \|Tx\|_X$ for all $x \in X_T$ [5, Lemma 2.1].

Let S and M be subsets of a metric space (X, d) and $\varepsilon > 0$. Then S is called an *ε-net* of M in X if for every $x \in M$ there exists $s \in S$ such that $d(x, s) < \varepsilon$. Further, if the set S is finite, then the ε-net S of M is called a *finite ε-net* of M, and we say that M has a finite ε-net in X. A subset of a metric space is said to be *totally bounded* if it has a finite ε-net for every $\varepsilon > 0$.

By $\mathcal{M}_X$, we denote the collection of all bounded subsets of a metric space (X, d). If $Q \in \mathcal{M}_X$, then the *Hausdorff measure of noncompactness* of the set Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\right\}.$$

The function $\chi : \mathcal{M}_X \to [0, \infty)$ is called the *Hausdorff measure of noncompactness* [4, p. 387].

The basic properties of the Hausdorff measure of noncompactness can be found in [6, Lemma 2]. For example, if Q, Q_1 and Q_2 are bounded subsets of a metric space (X, d), then we have

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is totally bounded,}$$
$$Q_1 \subset Q_2 \quad \text{implies } \chi(Q_1) \leq \chi(Q_2).$$

Further, if X is a normed space then the function χ has some additional properties connected with the linear structure, e.g.

$$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2),$$
$$\chi(\alpha Q) = |\alpha|\chi(Q) \quad \text{for all } \alpha \in \mathbb{C}.$$

We shall need the following known results for our investigation [7,6,3].

Lemma 1.1. *Let X denote any of the spaces c_0, c or ℓ_∞. Then, we have $X^\beta = \ell_1$ and $\|a\|_X^* = \|a\|_{\ell_1}$ for all $a \in \ell_1$.*

Lemma 1.2. *Let $X \supset \phi$ and Y be BK spaces. Then, we have $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ by $L_A(x) = Ax$ for all $x \in X$.*

Lemma 1.3. *Let $X \supset \phi$ be a BK space and Y be any of the spaces c_0, c or ℓ_∞. If $A \in (X, Y)$, then we have*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

Lemma 1.4. *Let T be a triangle. Then, we have*

(a) *For arbitrary subsets X and Y of w, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.*
(b) *Further, if X and Y are BK spaces and $A \in (X, Y_T)$, then $\|L_A\| = \|L_B\|$.*

Lemma 1.5. *Let $Q \in \mathcal{M}_{c_0}$ and $P_r : c_0 \to c_0$ $(r \in \mathbb{N})$ be the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in c_0$. Then, we have*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{x\in Q} \|(I - P_r)(x)\|_{\ell_\infty}\right),$$

where I is the identity operator on c_0.

Further, we know by [3, Theorem 1.10] that every $z = (z_n) \in c$ has a unique representation $z = \bar{z}e + \sum_{n=0}^{\infty}(z_n - \bar{z})e^{(n)}$, where $\bar{z} = \lim_{n\to\infty} z_n$. Thus, we define the projectors $P_r : c \to c (r \in \mathbb{N})$ by

$$P_r(z) = \bar{z}e + \sum_{n=0}^{r}(z_n - \bar{z})e^{(n)}; \quad (r \in \mathbb{N}) \tag{1.3}$$

for all $z = (z_n) \in c$ with $\bar{z} = \lim_{n\to\infty} z_n$. In this situation, the following result gives an estimate for the Hausdorff measure of noncompactness in the *BK* space c.

Lemma 1.6 ([6, *Theorem 5 (b)*]). *Let $Q \in \mathcal{M}_c$ and $P_r : c \to c$ $(r \in \mathbb{N})$ be the projector onto the linear span of $\{e, e^{(0)}, e^{(1)}, \ldots, e^{(r)}\}$. Then, we have*

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{x\in Q}\|(I - P_r)(x)\|_{\ell_\infty}\right) \le \chi(Q) \le \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I - P_r)(x)\|_{\ell_\infty}\right),$$

where I is the identity operator on c.

Moreover, we have the following result concerning with the Hausdorff measure of noncompactness in the matrix domains of triangles in normed sequence spaces.

Lemma 1.7 ([8, *Theorem 2.6*]). *Let X be a normed sequence space, T a triangle and χ_T and χ denote the Hausdorff measures of noncompactness on $\mathcal{M}_{X_T}$ and $\mathcal{M}_X$, the collections of all bounded sets in X_T and X, respectively. Then $\chi_T(Q) = \chi(T(Q))$ for all $Q \in \mathcal{M}_{X_T}$.*

The most effective way in the characterization of compact operators between the Banach spaces is by applying the Hausdorff measure of noncompactness. This can be achieved as follows (see [3, Theorem 2.25; Corollary 2.26]).

Let X and Y be Banach spaces. Then, the *Hausdorff measure of noncompactness* of L, denoted by $\|L\|_\chi$, is defined by

$$\|L\|_\chi = \chi(L(S_X)) \tag{1.4}$$

and we have

$$L \text{ is compact if and only if } \|L\|_\chi = 0. \tag{1.5}$$

2. Difference sequence spaces of weighted means

For a sequence $x = (x_k)$, we denote the difference sequence by $\Delta x = (x_k - x_{k-1})$. Let $u = (u_k)_{k=0}^{\infty}$ and $v = (v_k)_{k=0}^{\infty}$ be the sequences of real numbers such that $u_k \neq 0$, $v_k \neq 0$ for all $k \in \mathbb{N}$. Recently, the difference sequence spaces of weighted means $\lambda(u, v, \Delta)$ have been introduced in [9], where $\lambda = c, c_0$ and ℓ_∞. These sequence spaces are defined as the matrix domains of the triangle W in the spaces c, c_0 and ℓ_∞, respectively. The matrix $W = (w_{nk})$ is defined by

$$w_{nk} = \begin{cases} u_n(v_k - v_{k+1}); & (k < n), \\ u_n v_n; & (k = n), \\ 0; & (k > n). \end{cases} \tag{2.1}$$

It is obvious that $\lambda(u, v, \Delta)$ are *BK* spaces with the norm given by

$$\|x\|_\Lambda = \|W(x)\|_\infty = \sup_n |W_n(x)|. \tag{2.2}$$

Throughout, for any sequence $x = (x_k) \in w$, we define the *associated sequence* $y = (y_k)$, which will frequently be used, as the W-transform of x, that is $y = W(x)$. Then, it can easily be shown that

$$y_k = \sum_{j=0}^{k} u_k v_j \Delta x_j, \quad (k \in \mathbb{N}) \tag{2.3}$$

and hence,

$$x_k = \sum_{j=0}^{k} \frac{1}{v_j}\left(\frac{y_j}{u_j} - \frac{y_{j-1}}{u_{j-1}}\right); \quad (k \in \mathbb{N}). \tag{2.4}$$

If the sequences x and y are connected by the relation (2.3), then $x \in \lambda(u, v, \Delta)$ if and only if $y \in \lambda$; furthermore if $x \in \lambda(u, v, \Delta)$, then, $\|x\|_\Lambda = \|y\|_\infty$. In fact, since W is a triangle, the linear operator $L_W : X \to Y$, which maps every sequence in X to its associated sequence in Y, is bijective and norm preserving; where $X = \lambda(u, v, \Delta)$ and $Y = \lambda$.

For $u_n = \lambda_n$ and $v_k = \lambda_k - \lambda_{k-1}$ these spaces are reduced to the spaces studied in [10].

In [9, p. 4], the α-, β-, and γ-duals of the spaces $\lambda(u, v, \Delta)$ for $\lambda = c, c_0$ and ℓ_∞ have been determined and some related matrix transformations have also been characterized.

The following results will be needed in establishing our results.

Lemma 2.1. *Let X denote any of the spaces $c_0(u, v, \Delta)$ or $\ell_\infty(u, v, \Delta)$. If $a = (a_k) \in X^\beta$, then $\bar{a} = (\bar{a}_k) \in \ell_1$ and the equality*

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \bar{a}_k y_k \tag{2.5}$$

holds for every $x = (x_k) \in X$, where $y = W(x)$ is the associated sequence defined by (2.3) and

$$\bar{a}_k = \frac{1}{u_k}\left[\frac{a_k}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{\infty} a_j\right]; \quad (k \in \mathbb{N}).$$

Proof. This follows immediately by [10, Theorem 5.6]. □

Lemma 2.2. *Let X denote any of the spaces $c_0(u, v, \Delta)$ or $\ell_\infty(u, v, \Delta)$. Then, we have*

$$\|a\|_X^* = \|\bar{a}\|_{\ell_1} = \sum_{k=0}^{\infty} |\bar{a}_k| < \infty$$

for all $a = (a_k) \in X^\beta$, where $\bar{a} = (\bar{a}_k)$ is as in Lemma 2.1. □

Proof. Let Y be the respective one of the spaces c_0 or ℓ_∞, and take any $a = (a_k) \in X^\beta$. Then, we have by Lemma 2.1 that $\bar{a} = (\bar{a}_k) \in \ell_1$ and the equality (2.5) holds for all sequences $x = (x_k) \in X$ and $y = (y_k) \in Y$ which are connected by the relation (2.3). Further, it follows by (2.2) that $x \in S_X$ if and only if $y \in S_Y$. Therefore, we derive from (1.2) and (2.5) that

$$\|a\|_X^* = \sup_{x \in S_X}\left|\sum_{k=0}^{\infty} a_k x_k\right| = \sup_{y \in S_Y}\left|\sum_{k=0}^{\infty} \bar{a}_k y_k\right| = \|\bar{a}\|_Y^*$$

and since $\bar{a} \in \ell_1$, we obtain from Lemma 1.1 that

$$\|a\|_X^* = \|\bar{a}\|_Y^* = \|\bar{a}\|_{\ell_1} < \infty$$

which concludes the proof. □

Lemma 2.3. *Let X be any of the spaces $c_0(u, v, \Delta)$ or $\ell_\infty(u, v, \Delta)$, Y the respective one of the spaces c_0 or ℓ_∞, Z a sequence space and $A = (a_{nk})$ an infinite matrix. If $A \in (X, Z)$, then $\bar{A} \in (Y, Z)$ such that $Ax = \bar{A}y$ for all sequences $x \in X$ and $y \in Y$ which are connected by the relation (2.3), where $\bar{A} = (\bar{a}_{nk})$ is the associated matrix defined by*

$$\bar{a}_{nk} = \frac{1}{u_k}\left[\frac{a_{nk}}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{\infty} a_{nj}\right]; \quad (n, k \in \mathbb{N}) \tag{2.6}$$

provided the series on the right converge for all $n, k \in \mathbb{N}$.

Proof. Let $x \in X$ and $y \in Y$ be connected by the relation (2.3) and suppose that $A \in (X, Z)$. Then $A_n \in X^\beta$ for all $n \in \mathbb{N}$. Thus, it follows by Lemma 2.1 that $\bar{A}_n \in \ell_1 = Y^\beta$ for all $n \in \mathbb{N}$ and the equality $Ax = \bar{A}y$ holds, hence $\bar{A}y \in Z$. Further, we have by (2.4) that every $y \in Y$ is the associated sequence of some $x \in X$. Hence, we deduce that $\bar{A} \in (Y, Z)$. This completes the proof. □

Lemma 2.4. *Let X be any of the spaces $c_0(u, v, \Delta)$ or $\ell_\infty(u, v, \Delta)$, $A = (a_{nk})$ an infinite matrix and $\bar{A} = (\bar{a}_{nk})$ the associated matrix. If A is in any of the classes (X, c_0), (X, c) or (X, ℓ_∞), then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \left(\sum_{k=0}^{\infty} |\bar{a}_{nk}| \right) < \infty.$$

Proof. This is immediate by combining Lemmas 1.3 and 2.2. □

3. Compact operators on the spaces $c_0(u, v, \Delta)$ and $\ell_\infty(u, v, \Delta)$

In this section, we determine the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c_0(u, v, \Delta)$ or $\ell_\infty(u, v, \Delta)$, and apply our results to characterize some classes of compact operators on those spaces. For the most recent work on this topic, we refer to [11,12].

We begin with the following lemma [11, Lemma 3.1] which will be used in proving our results.

Lemma 3.1. *Let X denote any of the spaces c_0 or ℓ_∞. If $A \in (X, c)$, then we have*

$$\alpha_k = \lim_{n\to\infty} a_{nk} \quad \text{exists for every } k \in \mathbb{N},$$
$$\alpha = (\alpha_k) \in \ell_1,$$
$$\sup_n \left(\sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) < \infty,$$
$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \quad \text{for all } x = (x_k) \in X.$$

Now, let $A = (a_{nk})$ be an infinite matrix and $\bar{A} = (\bar{a}_{nk})$ the associated matrix defined by (2.6). Then, we have the following result on the Hausdorff measures of noncompactness.

Theorem 3.2. *Let X denote any of the spaces $c_0(u, v, \Delta)$ or $\ell_\infty(u, v, \Delta)$. Then, we have*

(a) *If $A \in (X, c_0)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\bar{a}_{nk}| \right). \tag{3.1}$$

(b) *If $A \in (X, c)$, then*

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\bar{a}_{nk} - \bar{\alpha}_k| \right) \le \|L_A\|_\chi \le \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\bar{a}_{nk} - \bar{\alpha}_k| \right), \tag{3.2}$$

where $\bar{\alpha}_k = \lim_{n\to\infty} \bar{a}_{nk}$ for all $k \in \mathbb{N}$.

(c) *If $A \in (X, \ell_\infty)$, then*

$$0 \le \|L_A\|_\chi \le \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\bar{a}_{nk}| \right). \tag{3.3}$$

Proof. Let us remark that the expressions in (3.1) and (3.3) exist by Lemma 2.4. Also, by combining Lemmas 2.3 and 3.1, we deduce that the expression in (3.2) exists.

We write $S = S_X$, for short. Then, we obtain by (1.4) and Lemma 1.2 that

$$\|L_A\|_\chi = \chi(AS). \tag{3.4}$$

For (a), we have $AS \in \mathcal{M}_{c_0}$. Thus, it follows by applying Lemma 1.5 that

$$\chi(AS) = \lim_{r\to\infty} \left(\sup_{x\in S} \|(I - P_r)(Ax)\|_{\ell_\infty} \right), \tag{3.5}$$

where $P_r : c_0 \to c_0 (r \in \mathbb{N})$ is the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in c_0$. This yields that $\|(I - P_r)(Ax)\|_{\ell_\infty} = \sup_{n>r} |A_n(x)|$ for all $x \in X$ and every $r \in \mathbb{N}$. Therefore, by using (1.1) and (1.2) and Lemma 2.2, we have for every $r \in \mathbb{N}$ that

$$\sup_{x\in S} \|(I - P_r)(Ax)\|_{\ell_\infty} = \sup_{n>r} \|A_n\|_X^* = \sup_{n>r} \|\bar{A}_n\|_{\ell_1}.$$

This and (3.5) imply that

$$\chi(AS) = \lim_{r\to\infty}\left(\sup_{n>r}\|\bar{A}_n\|_{\ell_1}\right) = \limsup_{n\to\infty}\|\bar{A}_n\|_{\ell_1}.$$

Hence, we get (3.1) by (3.4).

To prove (b), we have $AS \in \mathcal{M}_c$. Thus, we are going to apply Lemma 1.6 to get an estimate for the value of $\chi(AS)$ in (3.4). For this, let $P_r : c \to c (r \in \mathbb{N})$ be the projectors defined by (1.3). Then, we have for every $r \in \mathbb{N}$ that $(I - P_r)(z) = \sum_{n=r+1}^{\infty}(z_n - \bar{z})e^{(n)}$ and hence,

$$\|(I-P_r)(z)\|_{\ell_\infty} = \sup_{n>r}|z_n - \bar{z}| \tag{3.6}$$

for all $z = (z_n) \in c$ and every $r \in \mathbb{N}$, where $\bar{z} = \lim_{n\to\infty} z_n$ and I is the identity operator on c.

Now, by using (3.4), we obtain by applying Lemma 1.6 that

$$\frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right) \le \|L_A\|_\chi \le \lim_{r\to\infty}\left(\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\right). \tag{3.7}$$

On the other hand, it is given that $X = c_0(u, v, \Delta)$ or $\ell_\infty(u, v, \Delta)$, and let Y be the respective one of the spaces c_0 or ℓ_∞. Also, for every given $x \in X$, let $y \in Y$ be the associated sequence defined by (2.3). Since $A \in (X, c)$, we have by Lemma 2.3 that $\bar{A} \in (Y, c)$ and $Ax = \bar{A}y$. Further, it follows from Lemma 3.1 that the limits $\bar{\alpha}_k = \lim_{n\to\infty}\bar{a}_{nk}$ exist for all k, $\bar{\alpha} = (\bar{\alpha}_k) \in \ell_1 = Y^\beta$ and $\lim_{n\to\infty}\bar{A}_n(y) = \sum_{k=0}^{\infty}\bar{\alpha}_k y_k$. Consequently, we derive from (3.6) that

$$\begin{aligned}\|(I-P_r)(Ax)\|_{\ell_\infty} &= \|(I-P_r)(\bar{A}y)\|_{\ell_\infty}\\ &= \sup_{n>r}\left|\bar{A}_n(y) - \sum_{k=0}^{\infty}\bar{\alpha}_k y_k\right|\\ &= \sup_{n>r}\left|\sum_{k=0}^{\infty}(\bar{a}_{nk} - \bar{\alpha}_k)y_k\right|\end{aligned}$$

for all $r \in \mathbb{N}$. Moreover, since $x \in S = S_X$ if and only if $y \in S_Y$, we obtain by (1.2) and Lemma 1.1 that

$$\begin{aligned}\sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty} &= \sup_{n>r}\left(\sup_{y\in S_Y}\left|\sum_{k=0}^{\infty}(\bar{a}_{nk}-\bar{\alpha}_k)y_k\right|\right)\\ &= \sup_{n>r}\|\bar{A}_n - \bar{\alpha}\|_Y^*\\ &= \sup_{n>r}\|\bar{A}_n - \bar{\alpha}\|_{\ell_1}\end{aligned}$$

for all $r \in \mathbb{N}$. Hence, from (3.7) we get (3.2).

Finally, to prove (c) we define the operators $P_r : \ell_\infty \to \ell_\infty (r \in \mathbb{N})$ as in the proof of part (a) for all $x = (x_k) \in \ell_\infty$. Then, we have

$$AS \subset P_r(AS) + (I - P_r)(AS); \quad (r \in \mathbb{N}).$$

Thus, it follows by the elementary properties of the function χ that

$$\begin{aligned}0 &\le \chi(AS) \le \chi(P_r(AS)) + \chi((I-P_r)(AS))\\ &= \chi((I-P_r)(AS))\\ &\le \sup_{x\in S}\|(I-P_r)(Ax)\|_{\ell_\infty}\\ &= \sup_{n>r}\|\bar{A}_n\|_{\ell_1}\end{aligned}$$

for all $r \in \mathbb{N}$ and hence,

$$\begin{aligned}0 &\le \chi(AS) \le \lim_{r\to\infty}\left(\sup_{n>r}\|\bar{A}_n\|_{\ell_1}\right)\\ &= \limsup_{n\to\infty}\|\bar{A}_n\|_{\ell_1}.\end{aligned}$$

This and (3.4) together imply (3.3) and complete the proof. □

Corollary 3.3. *Let X denote any of the spaces $c_0(u, v, \Delta)$ or $\ell_\infty(u, v, \Delta)$. Then, we have*

(a) *If $A \in (X, c_0)$, then*

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\bar{a}_{nk}|\right) = 0.$$

(b) *If $A \in (X, c)$, then*

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\bar{a}_{nk} - \bar{\alpha}_k|\right) = 0,$$

where $\bar{\alpha}_k = \lim_{n\to\infty}\bar{a}_{nk}$ for all $k \in \mathbb{N}$.

(c) *If $A \in (X, \ell_\infty)$, then*

$$L_A \text{ is compact if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\bar{a}_{nk}|\right) = 0. \tag{3.8}$$

Proof. This result follows from Theorem 3.2 by using (1.5). □

It is worth mentioning that the condition in (3.8) is only a sufficient condition for the operator L_A to be compact, where $A \in (X, \ell_\infty)$ and X is any of the spaces $c_0(u, v, \Delta)$ or $\ell_\infty(u, v, \Delta)$. More precisely, the following example will show that it is possible for L_A to be compact while $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\bar{a}_{nk}|\right) \neq 0$. Hence, in general, we have just 'if' in (3.8) of Corollary 3.3(c).

Example 3.4. Let X be any of the spaces $c_0(u, v, \Delta)$ or $\ell_\infty(u, v, \Delta)$, and define the matrix $A = (a_{nk})$ by $a_{n0} = 1$ and $a_{nk} = 0$ for $k \geq 1 (n \in \mathbb{N})$. Then, we have $Ax = x_0 e$ for all $x = (x_k) \in X$, hence $A \in (X, \ell_\infty)$. Also, since L_A is of finite rank, L_A is compact. On the other hand, by using (2.6), it can easily be seen that $\bar{A} = A$. Thus $\bar{A}_n = e^{(0)}$ and so $\|\bar{A}_n\|_{\ell_1} = 1$ for all $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty}\|\bar{A}_n\|_{\ell_1} = 1$.

Finally, we have the following observation:

Corollary 3.5. *For every matrix $A \in (\ell_\infty(u, v, \Delta), c_0)$ or $A \in (\ell_\infty(u, v, \Delta), c)$, the operator L_A is compact.*

Proof. Let $A \in (\ell_\infty(u, v, \Delta), c_0)$. Then, we have by Lemma 2.3 that $\bar{A} \in (\ell_\infty, c_0)$ which implies that $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\bar{a}_{nk}|\right) = 0$ [9, p. 4]. This leads us with Corollary 3.3(a) to the consequence that L_A is compact. Similarly, if $A \in (\ell_\infty(u, v, \Delta), c)$ then $\bar{A} \in (\ell_\infty, c)$ and so $\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\bar{a}_{nk} - \bar{\alpha}_k|\right) = 0$, where $\bar{\alpha}_k = \lim_{n\to\infty}\bar{a}_{nk}$ for all k. Hence, we deduce from Corollary 3.3(b) that L_A is compact. □

References

[1] J. Boos, Classical and Modern Methods in Summability, Oxford University Press, New York, 2000.
[2] A. Wilansky, Summability Through Functional Analysis, in: North-Holland Mathematics Studies, vol. 85, Elsevier Science Publishers, Amsterdam, New York, Oxford, 1984.
[3] E. Malkowsky, V. Rakočević, An introduction into the theory of sequence spaces and measures of noncompactness, Zb. Rad. Mat. Inst. SANU 9 (17) (2000) 143–234.
[4] E. Malkowsky, V. Rakočević, S. Živković, Matrix transformations between the sequence spaces $w_0^p(\Lambda)$, $v_0^p(\Lambda)$, $c_0^p(\Lambda)(1 < p < \infty)$ and certain *BK* spaces, Appl. Math. Comput. 147 (2) (2004) 377–396.
[5] I. Djolović, E. Malkowsky, A note on compact operators on matrix domains, J. Math. Anal. Appl. 340 (1) (2008) 291–303.
[6] E. Malkowsky, Compact matrix operators between some *BK* spaces, in: M. Mursaleen (Ed.), Modern Methods of Analysis and its Applications, Anamaya Publ., New Delhi, 2010, pp. 86–120.
[7] B. de Malafosse, V. Rakočević, Applications of measure of noncompactness in operators on the spaces s_α, s_α^0, $s_\alpha^{(c)}$, ℓ_α^p, J. Math. Anal. Appl. 323 (1) (2006) 131–145.
[8] I. Djolović, Compact operators on the spaces $a_0^r(\Delta)$ and $a_c^r(\Delta)$, J. Math. Anal. Appl. 318 (2) (2006) 658–666.
[9] H. Polat, V. Karakaya, N. Simşek, Difference sequence spaces derived by using a generalized weighted mean, Appl. Math. Lett. 24 (2011) 608614.
[10] M. Mursaleen, A.K. Noman, On some new difference sequence spaces of non-absolute type, Math. Comput. Modelling 52 (2010) 603–617.
[11] M. Mursaleen, A.K. Noman, Applications of the Hausdorff measure of noncompactness in some sequence spaces of weighted means, Comput. Math. Appl. 60 (2010) 1245–1258.
[12] M. Mursaleen, A.K. Noman, Compactness by the Hausdorff measure of noncompactness, Nonlinear Anal. TMA 73 (2010) 2541–2557.